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Stochastic Modelling of 1-D Shallow Water Flows over Uncertain Topography

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A second-order perturbation approach is used to investigate the effects of topographic uncertainty on a numerical model of shallow water flow. The governing equation is discretised using finite differences, the resulting nonlinear system expanded as a Taylor series about the unperturbed water depth to first and second-order, and the resulting matrix equation solved to derive second-order moments for the model solution. A Fourier technique is used to estimate the accuracy of the first- and second-order approximations and indicates that for even small perturbations, second-order terms are significant. Results are compared to those from Monte Carlo simulations, showing that significant nonlinear effects are well represented by the second-order stochastic model, predicting correctly the shift in the mean depth and an increase in the depth variance. The statistics of the solution are however still well represented by a Gaussian distribution, and therefore moments greater than order 2 need not be calculated.

Key Words: stochastic differential equations; Monte Carlo methods; shallow water equations.

1. INTRODUCTION

The application of numerical modelling techniques to environmental problems is hampered by two significant problems. First, our knowledge of the natural environment is subject to significant levels of uncertainty, exacerbated by the spatial heterogeneity of many natural fields. Thus model parameters are often poorly known. Second, the equations governing environmental process tend to be nonlinear, making it difficult to map uncertainty in model parameters to uncertainty in model predictions. Furthermore, for cases where even the deterministic problem is computationally intensive, the incorporation of uncertainty (for example using Monte Carlo techniques) may lead to an intractable problem. We should therefore seek numerically efficient ways of transforming uncertain model parameters into uncertain model predictions.

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One approach to the problem of parameter uncertainty is the Monte Carlo method, whereby a large number of realisations of the random parameter field are generated and used as input to a deterministic model, and the resulting ensemble of model predictions is used to derive the statistics of the model solution. This approach has been adopted for both ground water [1–3] and free surface [4–7] flows and can be used with highly nonlinear (even chaotic) systems, where model response is far from a smooth function of model parameters. The disadvantage is a computational one: the technique is of exponential order in the number of model parameters and is thus impractical for all but the simplest of systems, where only a few parameters are subject to uncertainty. The direct application of the Monte Carlo technique to models parameterised by random fields (with effectively infinite degrees of freedom) may be unfeasible, and some finite subset of possible realisations of the field must be used and the results taken to be representative of the general problem. The Monte Carlo method does, however, give a conceptually simple, if computationally intensive, method of dealing with parameter uncertainty and deals implicitly with nonlinear model behaviour.

Another approach is to deal with uncertainty analytically, through perturbation techniques, which are often applied to groundwater flow problems [8–10] and to overland flow [11]. Uncertain parameters are decomposed into systematic and random components, and a Taylor expansion in the random component is used to estimate model response. This is a relatively efficient technique and has been applied up to second order in the random components, thus allowing some of the nonlinear behaviour of the system to be modelled. This technique has yet to be applied to models of shallow water flows with similar problems of uncertain parameter fields and nonlinear behaviour. As with any approximate technique, an important aspect is the development of criteria to assess the validity of the first- and second-order expansions used, so that the appropriate approximation can be used.

The research presented in this paper therefore aims to use first- and second-order perturbation techniques to develop a stochastic model of shallow water flow, using the bed topography as an example of an uncertain, spatially heterogeneous parameterising field. The stochastic model can then be tested against Monte Carlo simulations using a deterministic numerical model. Since the solutions are unknown, it is difficult to make *a priori* judgements concerning the accuracy of first- and second-order approximations, and some analytical estimate of the properties of the distributions of the model prediction statistics would be invaluable in selecting the appropriate level of approximation. To this end a Fourier analysis of the nonlinear problem is used to estimate solution uncertainty in a closed form, enabling the appropriate approximation to be made according to the flow regime and magnitude of parameter uncertainty. While this is a very simple model of free surface flow, it can be viewed as a first step towards uncertain 2-D and 3-D flow models, and towards developing a rationale for more complex modelling studies.

2. MODEL DEVELOPMENT

2.1. *Fourier Method*

Depth-averaged steady state shallow water flow in one dimension can be described by the diffusive wave approximation

$$\frac{dz}{dx} + \frac{dh}{dx} + \frac{q^2 n^2}{h^{10/3}} = 0, \tag{1}$$

where z is the bed elevation, h is the flow depth, q is the discharge per unit width, and n is Manning's friction coefficient. Writing the depth as the sum of an unperturbed value h_0 (the solution for the unperturbed bed elevation) and a perturbation h_1, expanding z similarly and developing a Taylor series for the nonlinear term, gives

$$\frac{dz_1}{dx} + \frac{dh_1}{dx} - \frac{10q^2n^2}{3h_0^{13/3}}h_1 + \frac{65q^2n^2}{9h_0^{16/3}}h_1^2 - \frac{1040q^2n^2}{81h_0^{19/3}}h_1^3 + O\left(h_1^4\right) = 0. \tag{2}$$

Zero-order terms cancel as they form the solution to the unperturbed problem. The equation in its linear form is given by

$$\frac{dz_1}{dx} + \frac{dh_1}{dx} - a_1 h_1 = 0, \tag{3}$$

where a_i denotes the coefficient of the ith term of the expansion. Equation 3 is easily solved via a Fourier transform, as long as the random processes describing the bed and depth perturbations are stationary. This gives expressions for the transform of the depth perturbation,

$$H_1 = \frac{-ikZ_1}{ik + a_1}, \tag{4}$$

and free surface perturbation $h_f (= h_1 + z_1)$,

$$H_f = \frac{a_1 Z_1}{ik + a_1}, \tag{5}$$

with uppercase letters denoting the Fourier transform of the equivalent lowercase variables and k the spatial wavenumber. The random perturbation fields are now completely defined in frequency space by their power spectra. The transformed perturbation will clearly depend on Z_1, which can be related to the covariance function of z_1 via the Wiener–Kintchine theorem

$$|Z_1|^2 = \int_{-\infty}^{\infty} B_z(r) e^{ikr} \, dr, \tag{6}$$

where $B_z(r)$ is the covariance of two points separated by distance r. The dependence of $B_z(r)$ on the spatial separation of the two points, and not on their absolute position, is a result of the stationarity of the process. Given an appropriate correlation model, e.g., the exponential model for $B_z(r)$,

$$B_z(r) = \sigma_z^2 e^{-\frac{|r|}{l}}, \tag{7}$$

where σ_z^2 is the variance of the process and l the correlation length, the power spectrum of the depth perturbations is given by

$$|Z_1|^2 = \frac{2l\sigma_z^2}{k^2l^2 + 1}. \tag{8}$$

The power spectra for the depth and free surface perturbations can then be used to derive the variance (covariance at zero lag) of the random depth and free surface perturbation fields (Wiener–Kintchine theorem again):

$$\sigma_h^2 = \frac{1}{2\pi} \int_{k=-\infty}^{k=+\infty} \frac{k^2}{k^2 + a_1^2} \frac{2l\sigma_z^2}{k^2l^2 + 1} dk = \frac{\sigma_z^2}{1 + a_1 l}, \tag{9}$$

$$\sigma_f^2 = \frac{1}{2\pi} \int_{k=-\infty}^{k=+\infty} \frac{a_1^2}{k^2 + a_1^2} \frac{2l\sigma_z^2}{k^2l^2 + 1} dk = \frac{\sigma_z^2 a_1 l}{1 + a_1 l}. \tag{10}$$

Equations 9 and 10 corroborate our intuitive understanding of free surface flows. For $a_1 l \ll 1$, the free surface perturbations will be much smaller than those of the bed, and the depth perturbations will be approximately equal in magnitude, and opposite in sign, to the bed perturbations. As the correlation length of the bed perturbations increases, the free surface perturbations also increase, until for $a_1 l \gg 1$ the free surface perturbations are equal to the bed perturbations. In this case variations in the depth become small. $1/a_1$ therefore defines a characteristic length scale for the flow; bed perturbations below this length tend to be smoothed out and have little effect on the free surface height, and above this length they influence the free surface height.

Having determined some statistical properties of the solution to the linearised problem, we can now estimate the conditions under which the first-order approximation is valid, and when higher order terms need to be included in the expansion. For example, including terms in h_1^2 in Eq. (2) and taking the expectation value will cause a nonzero mean in the depth perturbations:

$$\left\langle \frac{dz_1}{dx} \right\rangle + \left\langle \frac{dh_1}{dx} \right\rangle - a_1 \langle h_1 \rangle + a_2 \langle h_1^2 \rangle = 0. \tag{11}$$

If we assume that the statistics of z_1 and h_1 are stationary, the derivative terms vanish, and the variance of h_1 in the last term can be approximated by the expression derived from the linear solution:

$$\langle h_1 \rangle = \frac{a_2}{a_1} \frac{\sigma_z^2}{1 + a_1 l}. \tag{12}$$

Thus nonlinear effects will produce a positive shift in the mean value of the depth and the free surface. This effect may well be less than the standard deviation of the depth perturbation (dominated by a strong negative correlation with the topographic perturbations), but may be significant if the free surface elevation is of interest. This implies that the condition

$$\frac{\sigma_z}{h_0} \ll (a_1 l (1 + a_1 l))^{1/2} \tag{13}$$

must be met for the linear approximation to be valid. This condition is most likely to be violated for low flow rates, shallow flows (small a_1), or bed perturbations with short correlation lengths (rough topography). The validity of the second-order approximation can be tested by comparing the magnitude of the second- and third-order terms (again using the linear solution as an estimate of the depth perturbations), giving

$$\frac{\sigma_z}{h_0} \ll (1 + a_1 l)^{1/2}. \tag{14}$$

For most natural flows $a_1 l \ll 1$, which implies that third-order terms will be negligible until the magnitude of the bed perturbation approaches the flow depth. The inequality of Eq. (14) may still hold for even large bed perturbations (i.e., larger than the flow depth), provided their correlation length is long enough.

2.2. *Monte Carlo Model*

The Monte Carlo method uses a deterministic model to map, in a discrete fashion, the model input parameter space to the solution space. Thus the technique has two components: a generator for realisations of the input parameter process and a deterministic model to generate solutions for each realisation, both relying on a finite difference discretisation of the continuous case.

Before input parameter fields can be generated, some assumptions must be made about the statistics of the random field. In this case we assume that the field can be represented as a Gaussian process. Studies of the topography of gravel beds [12, 13] have shown this to be a reasonable assumption for these types of rivers, and Gaussian statistics have been used to describe the soil-covered landscape [14, 15]. The Gaussian random field can also be viewed as a worst case scenario in the maximum-entropy sense given if only second-order statistics are measured.

Given a continuous zero mean Gaussian random field $z(x)$, with covariance function given by the exponential model (7), the field is described in its discrete form (a vector $\mathbf{z}$ with n components z_i giving the values of the field at points separated by Δx) by its covariance matrix $\mathbf{B}$:

$$B_{ij} = \langle z_i z_j \rangle. \tag{15}$$

$\mathbf{B}$ is clearly symmetric, having n real eigenvalues and orthogonal eigenvectors, and can therefore be diagonalised using a transformation matrix $\mathbf{R}$ whose columns are made up of the eigenvectors of $\mathbf{B}$. Since a diagonal covariance matrix corresponds to a process of independent variables (which is easily generated), the original process can be written as

$$\mathbf{z} = \mathbf{R} \begin{bmatrix} N(0, \lambda^1) \\ N(0, \lambda^2) \\ \cdots \\ N(0, \lambda^n) \end{bmatrix}, \tag{16}$$

where $N(0, \lambda^i)$ is a zero mean Gaussian random variable with variance given by the ith eigenvalue of $\mathbf{B}$, λ^i.

The deterministic model uses a finite difference first-order accurate discretisation of (1),

$$\frac{h_{i+1} - h_i}{\Delta x} + \frac{z_{i+1} - z_i}{\Delta x} + q^2 n^2 h_i^{-10/3} = 0, \tag{17}$$

which results in a nonlinear system of equations which can be solved iteratively using the Newton–Raphson algorithm. A boundary condition is applied such that the free surface perturbation at the downstream end of the model domain is zero.

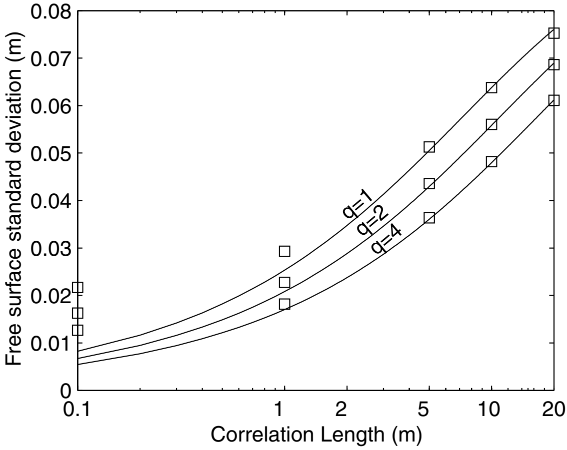

FIG. 1. Magnitude of free surface perturbations as a function of bed perturbation correlation length for three values of flow rate, with bed perturbations of 0.1 m. Values predicted by the Fourier method are shown as solid lines; Monte Carlo results are shown as squares.

Monte Carlo simulations were performed by adding the random perturbations generated by (16) to an unperturbed bed of uniform slope 10^{-2}, with the friction coefficient set at a typical value of 0.03. The domain was 200 m long and was discretised into 200 nodes 1 m apart. The effects of the downstream boundary condition (i.e., zero free surface perturbation) were eliminated by measuring statistics in the upstream half of the domain only. Figure 1 compares the free surface standard deviation derived from Monte Carlo simulations (10^4 realisations) and predicted by the Fourier analysis against bed perturbation correlation length for flow rates $q = 1, 2$ and 4 m^2 s^{-1}. Bed perturbations had a standard deviation of 0.1 m. There is a reasonable correspondence between the two techniques—except at short correlation lengths, which is not surprising given that the discrete model will be unable to distinguish between random fields with correlation lengths less than the grid spacing.

Figure 2 shows the effect of nonlinear terms on the solutions as the magnitude of the bed perturbations is increased. The shift in the free surface is predicted well by the Fourier method. The first-order estimates of the free surface variance are seen to be reasonable estimates of the variance measured in the Monte Carlo experiment, even when the shift in the free surface is significant. The criteria for the validity of the linear approximation (Eq. (13)) is also shown and corresponds to the point where the mean free surface elevation diverges significantly from its unperturbed value.

2.3. *Stochastic Model*

We now aim to reproduce the results above via a stochastic finite difference model that will avoid the need for a large ensemble of simulations and will also be able to cope with nonstationary fields and irregular unperturbed topography, thus combining the best features of the Monte Carlo and Fourier methods used above. Writing the finite difference formulation of Eq. (17) as a nonlinear vector equation,

$$\mathbf{f}(\mathbf{h}) = \mathbf{A}\mathbf{z}, \tag{18}$$

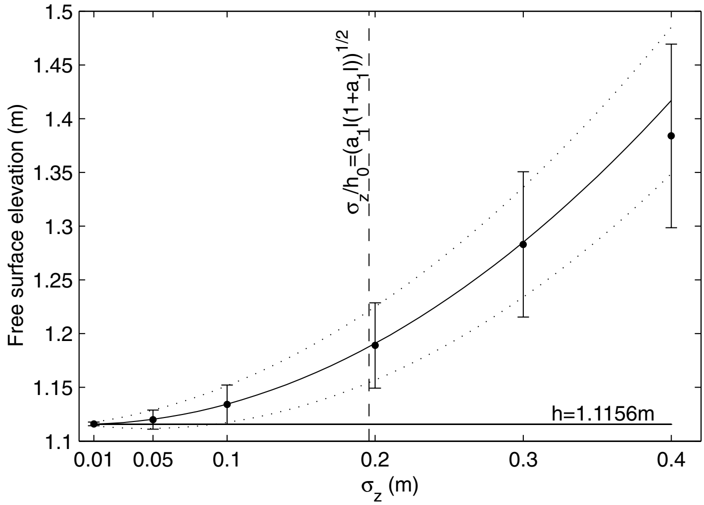

FIG. 2. Mean and variance of free surface elevation with increasing bed perturbation magnitude, shown for the Fourier method (solid and dashed lines) and the Monte Carlo method (error bars). The unperturbed elevation is shown at 1.1156 m, and the mean downstream slope has been removed. The criterion for the validity of the first-order approximation is $\sigma_z \ll 0.2$, agreeing with a significant shift in free surface height starting at ~ 0.1 m.

and again using a Taylor expansion to second order we obtain

$$\mathbf{f}(\mathbf{h}_0) + \mathbf{J}\mathbf{h}_1 + \frac{\mathbf{h}_1^T \underline{\mathbf{H}} \mathbf{h}_1}{2} + O\big(h_1^3\big) = \mathbf{A}(\mathbf{z}_0 + \mathbf{z}_1), \tag{19}$$

where vector quantities are denoted in bold and matrices by bold capital letters. $\mathbf{J}$ and $\underline{\mathbf{H}}$ are the Jacobian matrix and a vector of Hessian matrices, respectively, with elements given by

$$J_{ij} = \frac{\partial f_i}{\partial h_j} = \begin{bmatrix} -\frac{1}{\Delta x} - \frac{10a_1}{3h_0^{-13/3}} & \frac{1}{\Delta x} & 0 & \cdots \\ 0 & -\frac{1}{\Delta x} - \frac{10a_1}{3h_0^{-13/3}} & \frac{1}{\Delta x} & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & 0 & 1 \end{bmatrix}, \tag{20}$$

$$H_{jk}^i = \frac{\partial^2 f_i}{\partial h_j \partial h_k} = \frac{130 q^2 n^2}{9 h_i^{16/3}} \quad \text{for } i = j = k, 0 \text{ otherwise}. \tag{21}$$

If we assume that the solutions can be well approximated by Gaussian random fields (to be justified later), then second-order statistics will be sufficient to define those fields. The first-order solution is given by

$$\langle \mathbf{h}_1 \rangle = \mathbf{J}^{-1} \mathbf{A} \langle \mathbf{z}_1 \rangle = 0, \tag{22}$$

$$\big\langle \mathbf{h}_1 \mathbf{h}_1^T \big\rangle = \mathbf{J}^{-1} \mathbf{A} \big\langle \mathbf{z}_1 \mathbf{z}_1^T \big\rangle \mathbf{A}^T \mathbf{J}^{-1,T}, \tag{23}$$

and this gives a solution with zero mean, as before. The second-order approximations are made by adding a correction $\mathbf{h}_2$ and again substituting the linear solution into the nonlinear part:

$$\mathbf{J}(\mathbf{h}_1 + \mathbf{h}_2) + \mathbf{H}' diag\left(\mathbf{J}^{-1}\mathbf{A}\mathbf{z}_1\mathbf{z}_1^T\mathbf{A}^T\mathbf{J}^{-1,T}\right) = \mathbf{A}\mathbf{z}_1. \tag{24}$$

$\mathbf{H}'$ is the diagonal matrix derived from Eq. (21), and *diag* denotes a vector made up from the diagonal elements of the argument matrix. For the mean perturbation this gives the expression

$$\langle \mathbf{h}_2 \rangle = -\mathbf{J}^{-1}\mathbf{H}' diag\left\langle \mathbf{h}_1\mathbf{h}_1^T \right\rangle \tag{25}$$

and the covariance matrix

$$\left\langle \mathbf{h}_2\mathbf{h}_2^T \right\rangle = \mathbf{J}^{-1}\mathbf{H}'\langle \mathbf{K} \rangle \mathbf{H}'^T\mathbf{J}^{-1,T}, \tag{26}$$

where $\langle \mathbf{K} \rangle$ is a matrix of fourth-order moments with elements given by $\langle \mathbf{K} \rangle_{ij} = \langle h_{1,i}^2 h_{1,j}^2 \rangle$. Since $\mathbf{h}_2$ is a linear function of the covariance matrix of $\mathbf{h}_1$, the covariance of $\mathbf{h}_1$ and $\mathbf{h}_2$ will depend on the third moments of $\mathbf{h}_1$, which will be zero for the zero mean Gaussian process. Thus $\mathbf{h}_1$ and $\mathbf{h}_2$ are uncorrelated and their variances can be added directly to give the total variance of the depth perturbations. A similar argument implies that $\mathbf{h}_2$ and $\mathbf{z}_1$ are uncorrelated.

Figure 3 compares the behaviour of the second-order stochastic model with the Monte Carlo simulations and, as with the Fourier method, the shifts in the free surface are well predicted. However, it is difficult to assess whether the second-order predictions of the variance are an improvement over the first-order predictions. Figure 4 shows the probability distributions of the free surface as predicted by the Monte Carlo, Fourier, and second-order stochastic methods. The second-order stochastic model actually gives a much better fit to the distribution than is indicated by the variance alone—this is due to the significant tail

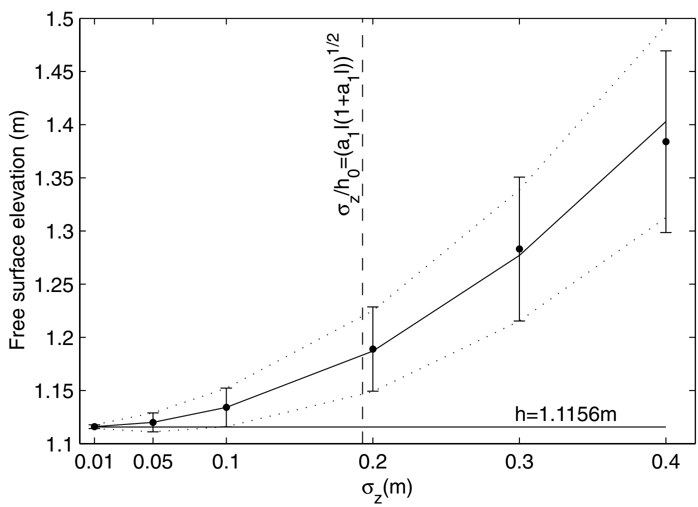

FIG. 3. Mean and variance of free surface elevation with increasing bed perturbation magnitude, shown for the second-order stochastic method (solid and dashed lines) and the Monte Carlo method (error bars). A performance similar to that of the Fourier technique is seen.

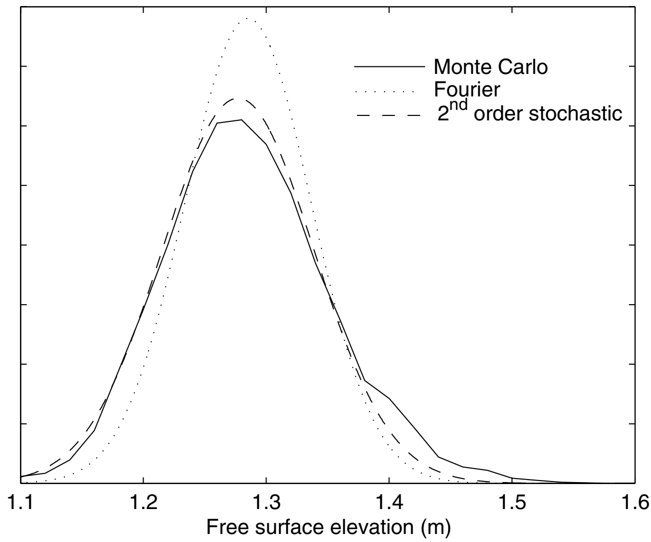

FIG. 4. Free surface height probability distributions predicted by Monte Carlo, Fourier, and second-order stochastic methods. The stochastic distribution is a good approximation to that given by the Monte Carlo method, apart from the tail at $h > 1.4$ m.

in the measured Monte Carlo distribution giving a higher estimate for the variance than the width of the distribution would suggest. The Gaussian distribution is, however, still a reasonable approximation to the measured distribution.

The main advantage of the stochastic finite difference scheme over the Monte Carlo method is its computational efficiency. Figure 5 shows the convergence of the mean free surface height and free surface standard deviation as a function of the number of Monte Carlo simulations. The expected $n^{-1/2}$ dependence is shown, with the uncertainty in the standard deviation being much greater than for the mean. The convergence rates indicate that somewhere between 10^3 and 10^4 simulations are required to achieve an estimate of the standard deviation as accurate as that given by the second-order stochastic model. Further, 10^4 Monte Carlo realisations and simulations required 50 s of processor (1.3 GHz Athlon) time, compared to 2 s for the second-order stochastic model, making the stochastic model an order of magnitude more efficient in this case.

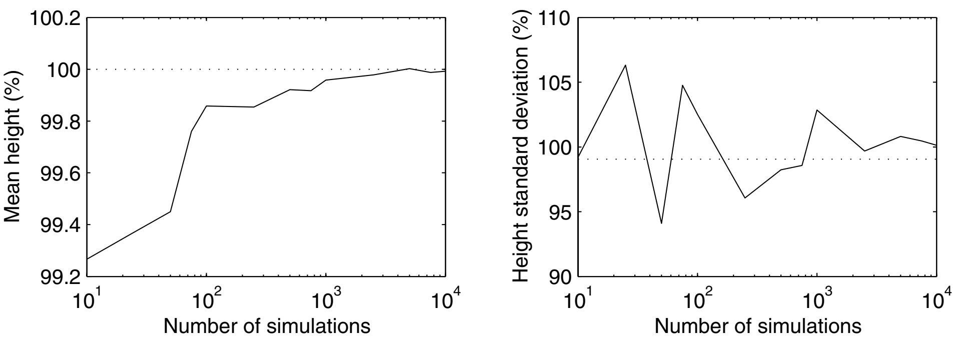

FIG. 5. Convergence of Monte Carlo method with increasing number of simulations.

3. EXTENSION TO 2-D FLOWS

Real shallow water flows, for example overbank flow in channel–floodplain systems or estuaries, are generally two dimensional in nature, with the governing equations

$$\nabla h + \nabla z + \frac{n^2 \mathbf{q}|\mathbf{q}|}{h^{10/3}} = 0, \tag{27}$$

$$\nabla \cdot \mathbf{q} = 0. \tag{28}$$

The discharge is now represented by the vector quantity $\mathbf{q}$. Flow velocities and depths are spatially heterogeneous, and any stochastic model of flow will have to be two dimensional to be of any practical use. It would be useful therefore to determine if the criteria for the validity of the first- and second-order expansions developed here can be applied to two-dimensional flows. The uncertainty in the nonlinear friction term comes from two sources: variations in depth and, for the two-dimensional case, variations in velocity. Variations in depth can be approximated by assuming that the free surface is much smoother than the bed topography (as demonstrated for the one-dimensional case in Eqs. (9) and (10)), and thus variations in depth are of the same magnitude as the bed perturbations. We would expect, therefore, the approximation to Eq. (14), $\sigma_z \ll h_0$, to hold, and thus the criteria for the validity of the second-order expansion of the friction term in h will be roughly the same for two-dimensional flows.

Estimating the magnitude of the variations in the discharge is more difficult, as the analytical solution of even the linearised problem is troublesome. We can, however, estimate the variations for two special cases and qualitatively extend the results to the general case. The situation shown on the left in Fig. 6, with topographic variations in the x-direction only, is essentially one dimensional, so the solutions developed by the Fourier method will be applicable, and the discharge per unit width will be uniform. The variance of $\mathbf{q}$ is therefore zero in this case. For topographic variations in the y-direction (right side of Fig. 6), the free surface will be of uniform slope S_0 in the x-direction, and the discharge per unit width will be given by

$$q_x(y) = \frac{S_0^{1/2}}{n}(h_{av} - z_1(y))^{5/3}, \tag{29}$$

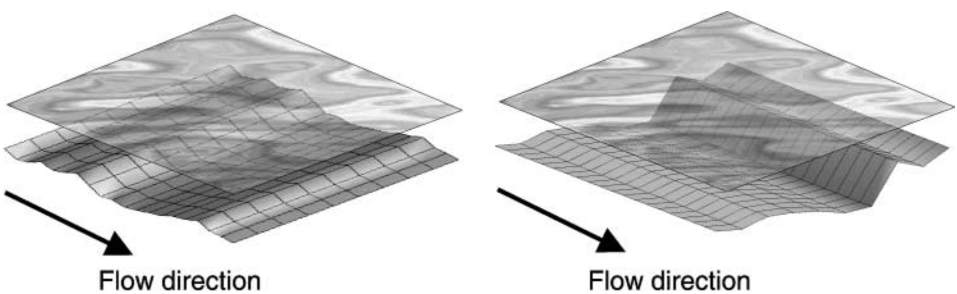

FIG. 6. (Left) Two-dimensional flow with ridges normal to the flow direction. Free surface perturbations are the same as the 1-D estimates. (Right) Ridges parallel to the flow direction. The free surface is planar, but discharge now varies in the direction normal to the flow.

where h_{av} is the mean depth. The standard deviation of q is then given approximately by

$$\sigma_q = \frac{5\sigma_z}{3h_0} q_0, \tag{30}$$

with σ_z the RMS bed perturbation and h_0 and q_0 the unperturbed depth and discharge. Thus the relative variance of q is of the same magnitude as the relative variance in the bed perturbations. We therefore speculate that a second-order approximation to the nonlinear friction term, if adequate for the treatment of bed perturbations, is also adequate for flow perturbations. Furthermore, since the dependence on $\mathbf{q}$ is analogous to a power law with exponent 2, and the depth dependence has exponent $10/3$, the second-order approximation for $\mathbf{q}|\mathbf{q}|$ should be more accurate than that for the depth. This is very much a "hand waving" argument, and the hypothesis that a second-order expansion is adequate to represent variations in the friction term due to the heterogeneous flow field needs to be tested using a fully two-dimensional model over an ensemble of Monte Carlo simulations.

4. CONCLUSIONS

First- and second-order perturbation approaches to the problem of modelling shallow water flows over uncertain topography have been developed, and model predictions compare favourably with those from the more computationally intensive Monte Carlo method. Fourier methods can also be used to estimate criteria for the use of first- and second-order stochastic models, which also agree well with numerical experiment. These criteria indicate that if the free surface elevation is of interest, a second-order expansion is required for even small bed perturbations, while a third-order expansion is not required until the magnitude of bed perturbations approaches the flow depth. Even when second-order effects are significant enough to produce sizeable shifts in the mean values of model predictions and increases in variance, the statistics of model predictions are still well described by Gaussian fields. The effect of the nonlinearity of the model is to effectively couple second-order moments in the bed perturbations to first-order moments in the free surface elevation. Thus good estimates of the bed roughness are required to estimate even the mean free surface elevation, and this will have consequences for the techniques used for model parameterisation.

The extension to 2-D flows has been explored in a qualitative fashion. The criteria for the application of first- and second-order expansions, developed using 1-D Fourier analysis, will still be applicable. We therefore have developed a rationale for the development of 2-D stochastic models of diffusive flow over uncertain topography, where second-order expansions will be adequate for approximating the statistics of model predictions. While the computational advantages of the stochastic method over the Monte Carlo method are trivial for the 1-D case, they should become a considerable advantage for 2-D models.

Future work should rigorously test whether the second-order perturbation approach is also valid for two-dimensional flows, given the variations the discharge bed perturbations will produce. This will allow the stochastic approach to be applied to real modelling problems, such as flood and estuarine flows, and the impact of topographic uncertainty on model predictions can then be assessed. Further work also needs to address the effects of uncertainty in other parameters, most notably the friction coefficient, the effect of using a more sophisticated process representation (e.g., inclusion of inertia and advection terms), and dynamic effects. Source terms arising from random lateral inflows may also be a significant source of uncertainty in model predictions.

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